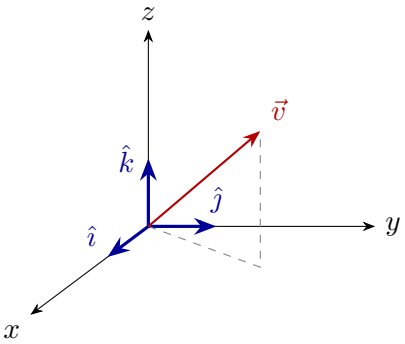
Chapter 0

Mathematical Preliminaries

Review chapter

This chapter reviews the vector and calculus tools used throughout the course and fixes the notation for vectors, unit vectors and derivatives. Note, this notation might be different than in other math courses you've taken, so review this carefully. It is reference material, not a lecture. Work through the review exercises at the end before the first lecture. If any of them feel unfamiliar, come and see me.

0.1 Vectors



A vector is written with an arrow, $\vec{v}$, and is specified by its components along the x, y and z axes:

$$\vec{v} = (v_x, v_y, v_z) = v_x\,\hat{\imath} + v_y\,\hat{\jmath} + v_z\,\hat{k} \tag{0.1}$$

The **basis vectors** $(\hat{\imath}, \hat{\jmath}, \hat{k})$ are unit vectors along the three axes. They are linearly independent: none of them can be written as a combination of the other two.

Addition and scalar multiplication

Vectors are added **component-wise**:

$$\vec{A} + \vec{B} = (A_x + B_x)\,\hat{\imath} + (A_y + B_y)\,\hat{\jmath} + (A_z + B_z)\,\hat{k}.$$

Multiplying by a scalar s multiplies every component:

$$s\vec{A} = sA_x\,\hat{\imath} + sA_y\,\hat{\jmath} + sA_z\,\hat{k}.$$

The dot product

The basis vectors satisfy

$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1, \qquad \hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{\imath}\cdot\hat{k} = 0,$$

so, for general vectors,

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \tag{0.2}$$

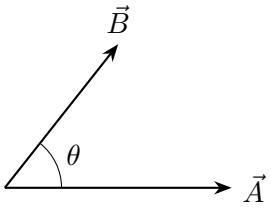
The dot product is a scalar, and it is symmetric: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$.

The **norm** (length) of a vector is

$$|\vec{A}| = A = \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

In terms of the lengths and the angle θ between the two vectors,

$$\boxed{\vec{A} \cdot \vec{B} = AB \cos\theta} \qquad (0.3)$$



In particular, $\vec{A} \cdot \vec{B} = 0$ when $\vec{A}$ and $\vec{B}$ are perpendicular.

Unit vectors

A hat denotes a vector of length one. Any non-zero vector $\vec{A}$ defines a unit vector in its direction,

$$\boxed{\hat{A} = \frac{\vec{A}}{|\vec{A}|}}$$

so that $\vec{A} = |\vec{A}|\,\hat{A}$: a length times a direction.

The cross product

The cross product of two vectors is another vector, most easily remembered as a determinant:

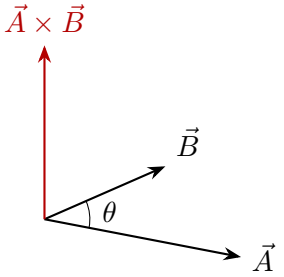
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\,\hat{\imath} + (A_z B_x - A_x B_z)\,\hat{\jmath} + (A_x B_y - A_y B_x)\,\hat{k}.$$

It is **antisymmetric**: $\vec{A} \times \vec{B} = -\,\vec{B} \times \vec{A}$.

Its length is

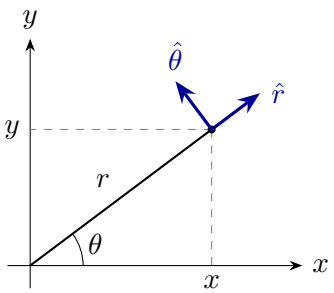
$$\boxed{|\vec{A} \times \vec{B}| = AB \sin\theta} \qquad (0.4)$$

and its direction is perpendicular to both $\vec{A}$ and $\vec{B}$, given by the **right-hand rule**: point the fingers of your right hand along $\vec{A}$ and curl them towards $\vec{B}$; your thumb points along $\vec{A} \times \vec{B}$.



0.2 Coordinate systems

Polar coordinates



A point (x, y) in the plane can instead be labelled by its distance r from the origin and the angle θ from the x-axis, $(x, y) \to (r, \theta)$:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The position vector is

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} = r\,\hat{r},$$

where $\hat{r}$ points away from the origin and $\hat{\theta}$ points in the direction of increasing θ.

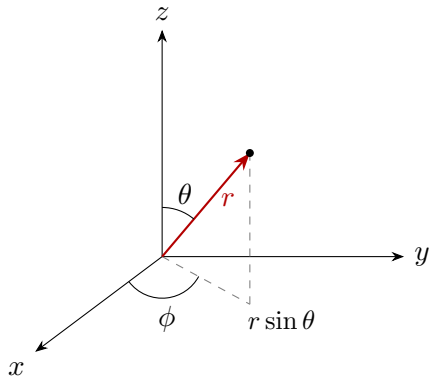
The area element is

$$\boxed{dA = dx\,dy = r\,dr\,d\theta}$$

Cylindrical coordinates

Add z to polar coordinates: $(x, y, z) \to (r, \theta, z)$, with $dV = r\,dr\,d\theta\,dz$.

Spherical coordinates



A point in space is labelled by its distance r from the origin, the angle θ from the z-axis, and the angle ϕ of its projection onto the xy-plane, measured from the x-axis:

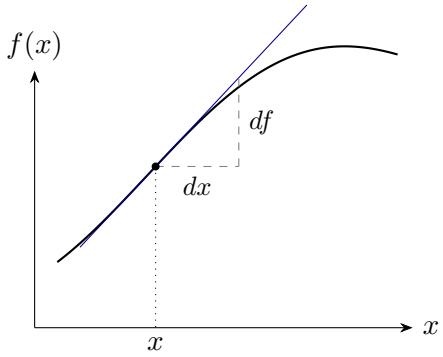
$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\dfrac{\sqrt{x^2 + y^2}}{z}\right) \\ \phi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{cases}$$

The projection onto the xy-plane has length $r \sin\theta$, so

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}$$

$$\boxed{dV = dx\,dy\,dz = r^2 \sin\theta\,dr\,d\theta\,d\phi}$$

0.3 Derivatives



The derivative of $f(x)$ is

$$\boxed{f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}} \tag{0.5}$$

Geometrically, it is the slope of the **tangent** to the graph of f at x.

Notation

A prime denotes a derivative with respect to x, and a dot a derivative with respect to time t:

$$\boxed{\frac{df}{dx} = f', \qquad \frac{dx}{dt} = \dot{x}, \qquad \frac{d^2x}{dt^2} = \ddot{x}}$$

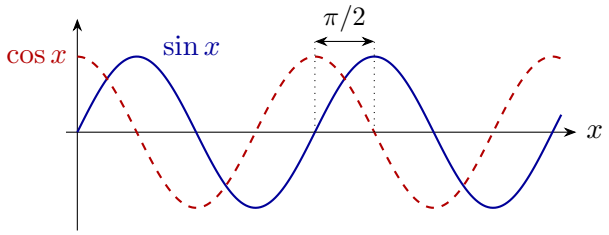
Standard derivatives

- The **power rule**:

$$\frac{d}{dx}\, x^n = n\, x^{n-1}.$$

- Sine and cosine:

$$\frac{d}{dx} \sin x = \cos x, \qquad \frac{d}{dx} \cos x = -\sin x.$$



Since $\cos x = \sin\left(x + \frac{\pi}{2}\right)$, differentiating a sine or a cosine shifts it to the left by a quarter period:

$$\frac{d}{dx} \iff \frac{\pi}{2} \text{ phase shift}$$

- The **product rule**:

$$\frac{d}{dx}(fg) = f'g + fg'.$$

- The **chain rule**:

$$\frac{d}{dx} f\big(g(x)\big) = \frac{dg}{dx}\frac{df}{dg}$$

Example: $\tan x$

Write $\tan x = \sin x \cdot \dfrac{1}{\cos x}$ and use the product rule, then the chain rule for $1/\cos x$:

$$\frac{d}{dx}\tan x = \frac{\cos x}{\cos x} + \sin x \frac{d}{dx}\left(\frac{1}{\cos x}\right) = 1 + \sin x \cdot \frac{\sin x}{\cos^2 x} = 1 + \tan^2 x.$$

Since $1 + \tan^2 x = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x}$,

$$\frac{d}{dx}\tan x = 1 + \tan^2 x = \sec^2 x.$$

The derivatives of the other trigonometric functions follow in the same way.

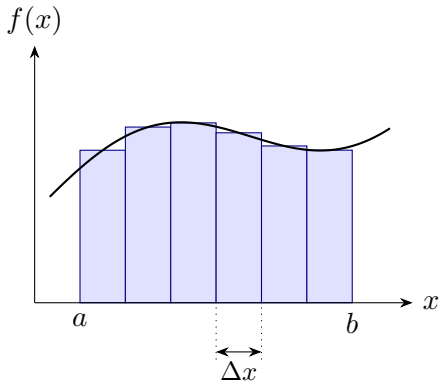
Exponentials and logarithms

$$\frac{d}{dx} e^x = e^x \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

Together with the rules above, these are all you need. For example, writing $a^x = e^{x \ln a}$ and using the chain rule,

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = \ln a \; e^{x \ln a} = a^x \ln a.$$

0.4 Integrals



The integral of $f(x)$ from a to b is the area under its graph. Cut $[a, b]$ into N strips of width $\Delta x = (b-a)/N$ and add up the areas of the rectangles:

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\,\Delta x \tag{0.6}$$

where x_i is a point in the i-th strip.

The area is *signed*: where $f(x) < 0$, the rectangles count negatively.

The fundamental theorem of calculus

A function F is an **antiderivative** of f if $F' = f$. The **indefinite integral** is the most general antiderivative,

$$\int f(x)\,dx = F(x) + C,$$

with an arbitrary constant C (its derivative is zero). The **fundamental theorem of calculus** says that integration undoes differentiation:

$$\boxed{\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)} \tag{0.7}$$

So every derivative you know, read backwards, gives an integral.

Polynomials

Reading the power rule backwards,

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C} \qquad (n \neq -1), \qquad\qquad \int \frac{dx}{x} = \ln|x| + C.$$

Integrals are linear, so a polynomial is integrated term by term. For example,

$$\int_0^2 \left(3x^2 - 2x + 1\right) dx = \Big[x^3 - x^2 + x\Big]_0^2 = (8 - 4 + 2) - 0 = 6.$$

Sine and cosine

Reading the derivatives of sine and cosine backwards,

$$\boxed{\int \sin x\,dx = -\cos x + C} \qquad\qquad \boxed{\int \cos x\,dx = \sin x + C}$$

For example, over half a period

$$\int_0^{\pi} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi} = -(-1) - (-1) = 2,$$

but over a full period the positive and negative areas cancel:

$$\int_0^{2\pi} \sin x\,dx = \Big[-\cos x\Big]_0^{2\pi} = -1 - (-1) = 0.$$

When the argument is ωt, the chain rule brings out a factor of ω, which the integral must divide back out:

$$\int \cos(\omega t)\,dt = \frac{1}{\omega}\sin(\omega t) + C, \qquad \int \sin(\omega t)\,dt = -\frac{1}{\omega}\cos(\omega t) + C.$$

Check by differentiating!

Problems

Problems are taken from the sources listed under Sources and Attribution. Numbers refer to the end-of-chapter problems; figures have been redrawn.

OpenStax Calculus Vol. 3, Chapter 2: Vectors in Space

2.427 Use the given vectors to find the quantities: $\vec{a} = 9\,\hat{\imath} - 2\,\hat{\jmath}$, $\vec{b} = -3\,\hat{\imath} + \hat{\jmath}$. (a) $3\vec{a} + \vec{b}$; (b) $|\vec{a}|$; (c) $\vec{a} \times |\vec{b} \times \vec{c}|$; (d) $\vec{b} \cdot \vec{a}$.

[Answer →]

2.429 Find the values of a such that the vectors $(2, 4, a)$ and $(0, -1, a)$ are orthogonal.

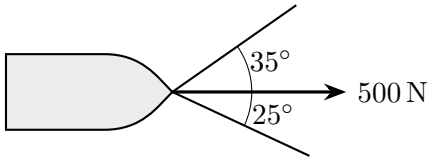
[Answer →]

2.443 Convert the given equation from cylindrical coordinates to rectangular coordinates. Identify the given surface.

$$r^2 - 2r\cos\theta + z^2 = 1$$

[Answer →]

2.445 A small boat is crossing a river. When the boat reaches the shore, two ropes are thrown to people to help pull the boat ashore. One rope is at an angle of 25° and the other is at 35°. If the boat must be pulled straight and at a force of 500 N, find the magnitude of force for each rope (see the figure).



[Answer →]

OpenStax Calculus Vol. 1, Chapter 3: Derivatives

Find the derivatives of the following functions.

3.373 $f(x) = 3x^3 - \dfrac{4}{x^2}$

[Answer →]

3.375 $f(x) = e^{\sin x}$

[Answer →]

3.377 $f(x) = x^2 \cos x + x \tan(x)$

[Answer →]

3.381 First derivative of $y = x \ln(x) \cos x$.

[Answer →]

OpenStax Calculus Vol. 1, Chapter 5: Integration

5.447 Evaluate the integral $\displaystyle\int_{-1}^{1} \left(x^3 - 2x^2 + 4x\right) dx$.

[Answer →]

5.449 Evaluate the integral $\displaystyle\int_{\pi/3}^{\pi/2} 2\sec(2\theta)\tan(2\theta)\, d\theta$.

[Answer →]

5.451 Find the antiderivative $\displaystyle\int \frac{dx}{(x+4)^3}$.

[Answer →]

Answers

Chapter 0: Mathematical Preliminaries

2.427 (a) $24\,\hat{\imath} - 5\,\hat{\jmath}$; (b) $\sqrt{85}$; (c) not defined (you can't cross a vector with a scalar); (d) -29 [← Problem]

2.429 $a = \pm 2$ [← Problem]

2.443 $x^2 - 2x + y^2 + z^2 = 1$, a sphere [← Problem]

2.445 331 N and 244 N [← Problem]

3.373 $f'(x) = 9x^2 + \dfrac{8}{x^3}$ [← Problem]

3.375 $f'(x) = e^{\sin x}\cos x$ [← Problem]

3.377 $f'(x) = x\sec^2(x) + 2x\cos(x) + \tan(x) - x^2\sin(x)$ [← Problem]

3.381 $y' = \cos x\,(\ln x + 1) - x\ln(x)\sin x$ [← Problem]

5.447 $-\dfrac{4}{3}$ [← Problem]

5.449 1 [← Problem]

5.451 $-\dfrac{1}{2(x+4)^2} + C$ [← Problem]

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The end-of-chapter problems in these notes are taken from the following openly licensed textbooks. Problem numbers in the text (for example 5.53) are the numbers of the end-of-chapter problems in the book named in the problems section.

- OpenStax, *University Physics Volume 2*, by Samuel J. Ling, Jeff Sanny and William Moebs. Licensed under CC BY 4.0. https://openstax.org/details/books/university-physics-volume-2
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- OpenStax, *Calculus Volume 3*, by Gilbert Strang, Edwin "Jed" Herman et al. Licensed under CC BY 4.0. https://openstax.org/details/books/calculus-volume-3

The signal-processing chapter also draws on the following book, which is not openly licensed and is cited rather than reproduced: Marina Bosi and Richard E. Goldberg, *Introduction to Digital Audio Coding and Standards*, Kluwer Academic Publishers, 2003. The approximation of the threshold in quiet used there is taken from that book; the plot of it is our own.

The Fourier problems at the end of the Fourier chapter were written for this course by the author.

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