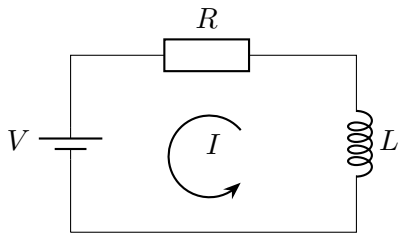
Chapter 5

RL, RLC and AC Circuits

5.1 RL circuits

Recall that a changing current through an inductor induces a voltage that opposes the change, $\mathcal{E}_L = -L\,dI/dt$ (3.6), where the **inductance** L depends only on geometry.



Consider a voltage source V, a resistor R and an inductor L in series. Because of the definition of inductance, the inductor contributes $\mathcal{E}_L = -L\,dI/dt$ to the loop rule:

$$V - IR - L\frac{dI}{dt} = 0 \tag{5.1}$$

This is the same equation as for the RC circuit, with $Q \leftrightarrow I$. Recall (2.2), the loop rule $V - IR - Q/C = 0$ with $I = dQ/dt$:

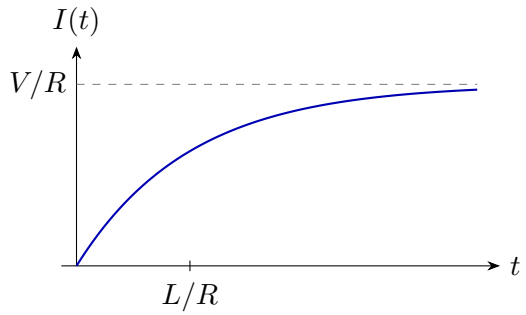
$$V - \frac{Q}{C} - R\frac{dQ}{dt} = 0, \qquad Q(t) = VC\left(1 - e^{-t/RC}\right).$$

Under the replacements

$$Q \to I, \qquad R \to L, \qquad C = \frac{1}{R}$$

this becomes (5.1), so

$$I(t) = \frac{V}{R}\left(1 - e^{-(R/L)\,t}\right)$$



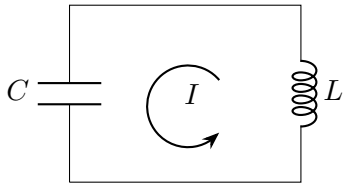
The time constant is L/R (compare RC). The inductor opposes changes in the current: I does not jump to V/R but grows gradually.

Connection with the oscillations chapter

We can also solve (5.1) with the strategy of the Oscillations chapter. The homogeneous equation $L\,dI/dt + RI = 0$ is solved by guessing $I = I_0\,e^{\lambda t}$, which gives $\lambda = -R/L$. A constant $I = V/R$

solves the full equation, so $I(t) = V/R + I_0\, e^{-(R/L)t}$, and $I(0) = 0$ fixes $I_0 = -V/R$: the same result. This split into a *homogeneous* and a *particular* solution reappears for the driven RLC circuit below.

5.2 LC circuits



With no source, the loop rule is $-L\, dI/dt - Q/C = 0$. Using $I = dQ/dt$,

$$\boxed{\frac{d^2Q}{dt^2} = -\frac{Q}{LC}} \tag{5.2}$$

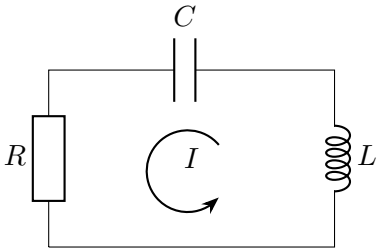
This is a harmonic oscillator!

Compare (4.1): Q plays the role of x and $1/LC$ the role of k/m. By (4.4),

$$\boxed{Q(t) = Q_0 \cos(\omega t + \phi)}\,, \qquad \boxed{\omega = \frac{1}{\sqrt{LC}}}$$

where ω is the **natural angular frequency** of the circuit.

5.3 RLC circuits



Adding a resistor to the LC circuit, the loop rule becomes

$$\boxed{\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0} \tag{5.3}$$

This is a damped oscillator!

Connection with the damped oscillator

Recall (4.5), $m\,\ddot{x} + b\,\dot{x} + kx = 0$. Equation (5.3) is the same equation with

$$m = 1, \qquad b = \frac{R}{L}, \qquad k = \frac{1}{LC}, \qquad x \leftrightarrow Q.$$

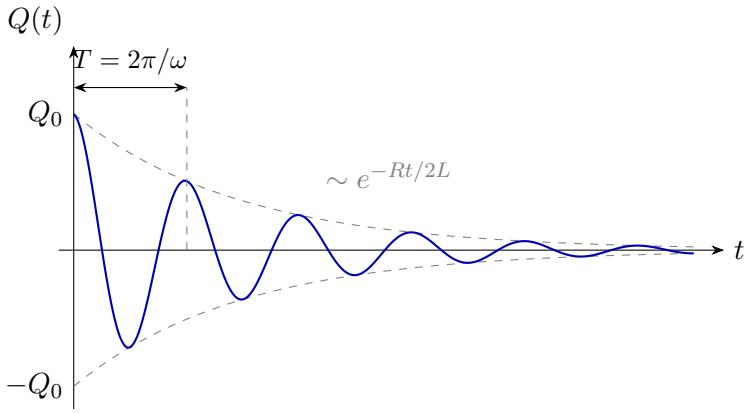
(Multiplying (5.3) by L gives the equivalent dictionary: inductance $\leftrightarrow$ mass, resistance $\leftrightarrow$ damping coefficient b, $1/C \leftrightarrow$ spring constant k.) The decay time $\tau = 2m/b$ becomes $2L/R$, and $\omega_0 = \sqrt{k/m}$ becomes the LC frequency $1/\sqrt{LC}$.

The solution of the damped oscillator then gives, in the underdamped case $1/LC > (R/2L)^2$,

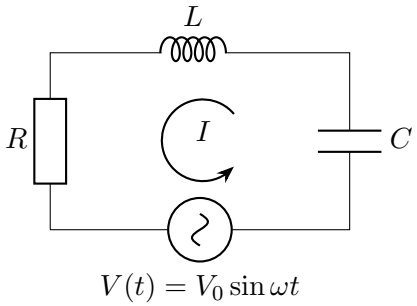
$$\boxed{Q(t) = Q_0\, e^{-(R/2L)\,t} \cos(\omega t + \phi)}$$

with

$$\boxed{\omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}}\,, \qquad T = \frac{2\pi}{\omega}.$$



5.4 Driven RLC circuit



Now add a source $V(t) = V_0 \sin \omega t$. The loop rule (K2) gives

$$V_0 \sin \omega t - L \frac{d^2Q}{dt^2} - R \frac{dQ}{dt} - \frac{Q}{C} = 0 \tag{5.4}$$

Strategy: homogeneous plus particular solution

Solve for $Q = Q_h + Q_p$ (the general solution). The homogeneous part Q_h solves the equation without the source, $L\ddot{Q}_h + R\dot{Q}_h + Q_h/C = 0$, so by the previous section

$$Q_h(t) = Q_0\, e^{-(R/2L)\,t} \cos(\omega' t + \phi), \qquad \omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}.$$

This decays exponentially, so at late times only the particular solution Q_p remains.

Finding Q_p

Guess a response at the driving frequency,

$$Q_p = \frac{A}{\omega} \cos \omega t + \frac{B}{\omega} \sin \omega t.$$

Then the current we are looking for is

$$I_p = \frac{dQ_p}{dt} = -A \sin \omega t + B \cos \omega t, \qquad \frac{d^2Q_p}{dt^2} = -A\omega \cos \omega t - \omega B \sin \omega t.$$

Plugging into (5.4) gives

$$\begin{aligned} V_0 \sin \omega t = {} & L\,(-A\omega \cos \omega t - B\omega \sin \omega t) \\ & + R\,(-A \sin \omega t + B \cos \omega t) \\ & + \frac{1}{\omega C}\,(A \cos \omega t + B \sin \omega t) \end{aligned}$$

Define

$$R_{LC} := \omega L - \frac{1}{\omega C}.$$

Matching the terms in $\cos\omega t$:

$$-L\omega A + RB + \frac{1}{\omega C}A = 0 \iff -R_{LC}A + RB = 0 \implies \boxed{A = \frac{R}{R_{LC}}B}$$

and in $\sin\omega t$:

$$-L\omega B - RA + \frac{1}{\omega C}B = V_0 \iff -R_{LC}B - RA = V_0 \implies \left(R_{LC} + \frac{R^2}{R_{LC}}\right)B = -V_0.$$

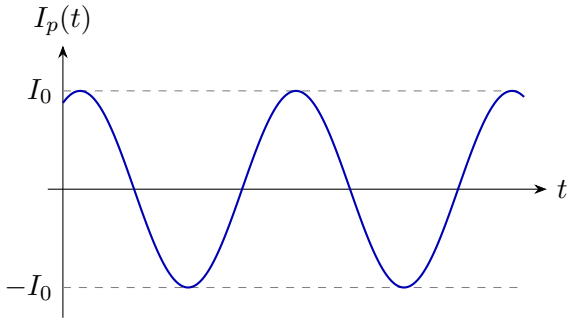
Hence

$$\boxed{B = -V_0\,\frac{R_{LC}}{R_{LC}^2 + R^2}} \qquad \boxed{A = -V_0\,\frac{R}{R_{LC}^2 + R^2}}$$

Amplitude and phase of the current

The particular current $I_p = B\cos\omega t - A\sin\omega t$ has the form $C\cos\omega t + D\sin\omega t$ of the Oscillations chapter, so by (4.4)

$$I_p(t) = I_0\cos(\omega t + \phi), \qquad I_0 = \sqrt{A^2 + B^2}, \qquad \tan\phi = \frac{A}{B}.$$



The current oscillates at the driving frequency ω, with the amplitude I_0 found next.

Substituting A and B,

$$\boxed{I_0 = V_0\sqrt{\frac{R_{LC}^2 + R^2}{\left(R_{LC}^2 + R^2\right)^2}} = \frac{V_0}{\sqrt{R_{LC}^2 + R^2}}}$$

Equivalently $I_0 = V_0/R_{\text{eff}}$ with the effective resistance

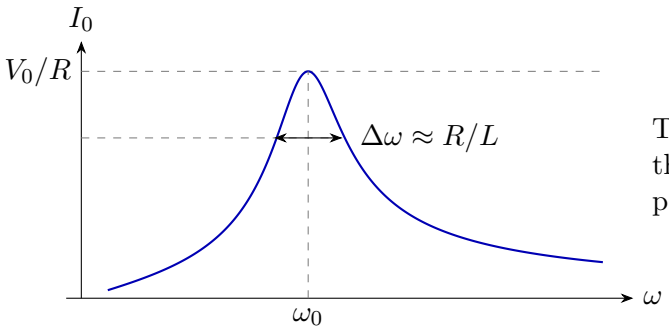
$$R_{\text{eff}} = \sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}.$$

Resonance

The amplitude I_0 is largest when $R_{LC} = \omega L - 1/(\omega C) = 0$, that is at

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$$

the natural frequency of the LC circuit, where $I_0 = V_0/R$. This is **resonance**!



The width of the resonance peak is $\Delta\omega \approx R/L$: the smaller the resistance, the sharper the peak.

Quality factor

The **quality factor** Q (not to be confused with the charge Q) measures how narrow the resonance is,

$$\boxed{\mathsf{Q} = \frac{\omega_0}{\Delta\omega}}$$

where $\Delta\omega$ is the width of the peak between the points where the dissipated power ($\propto I_0^2$) has dropped to half its maximum, i.e. where $I_0 = I_{\max}/\sqrt{2}$, equivalently $|R_{LC}| = R$. When the peak is narrow (small R), we can expand around ω_0:

$$R_{LC} = \frac{L\left(\omega^2 - \omega_0^2\right)}{\omega} \approx 2L\left(\omega - \omega_0\right),$$

so $|R_{LC}| = R$ at $|\omega - \omega_0| = R/2L$, a full width $\Delta\omega \approx R/L$. Therefore

$$\boxed{\mathsf{Q} \approx \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}}$$

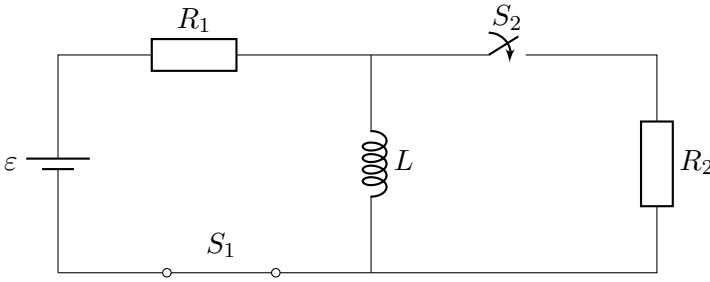
Using $\tau = 2L/R$ from the damped oscillator, this is also $\mathsf{Q} \approx \omega_0\tau/2$: a large Q means weak damping and a sharp resonance.

Problems

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OpenStax University Physics Vol. 2, Chapter 14: Inductance

14.53 For the circuit shown, $\varepsilon = 20\,\text{V}$, $L = 4.0\,\text{mH}$, and $R = 5.0\,\Omega$. After steady state is reached with S_1 closed and S_2 open, S_2 is closed and immediately thereafter (at $t = 0$) S_1 is opened. Determine (a) the current through L at $t = 0$, (b) the current through L at $t = 4.0 \times 10^{-4}\,\text{s}$, and (c) the voltages across L and R_2 at $t = 4.0 \times 10^{-4}\,\text{s}$. $R_1 = R_2 = R$.



[Answer →]

14.65 In an oscillating LC circuit, the maximum charge on the capacitor is $2.0 \times 10^{-6}\,\text{C}$ and the maximum current through the inductor is 8.0 mA. (a) What is the period of the oscillations? (b) How much time elapses between an instant when the capacitor is uncharged and the next instant when it is fully charged?

[Answer →]

14.71 In an oscillating RLC circuit with $L = 10\,\text{mH}$, $C = 1.5\,\mu\text{F}$, and $R = 2.0\,\Omega$, how much time elapses before the amplitude of the oscillations drops to half its initial value?

[Answer →]

OpenStax University Physics Vol. 2, Chapter 15: Alternating-Current Circuits

15.39 For an RLC series circuit, $R = 100\,\Omega$, $L = 150\,\text{mH}$, and $C = 0.25\,\mu\text{F}$. (a) If an ac source of variable frequency is connected to the circuit, at what frequency is maximum power dissipated in the resistor? (b) What is the quality factor of the circuit?

[Answer →]

15.41 (a) What is the resonant frequency of a resistor, capacitor, and inductor connected in series if $R = 100\,\Omega$, $L = 2.0\,\text{H}$, and $C = 5.0\,\mu\text{F}$? (c) What is the Q of the circuit? (d) What is the bandwidth of the circuit?

[Answer →]

Answers

Chapter 5: RL, RLC and AC Circuits

14.53 (a) 4.0 A; (b) 2.4 A; (c) on R_2: 12 V; on L: 12 V [← Problem]

14.65 (a) 1.6×10^{-3} s; (b) 3.9×10^{-4} s [← Problem]

14.71 6.9 ms [← Problem]

15.39 (a) 820 Hz; (b) 7.8 [← Problem]

15.41 (a) 50 Hz; (c) 6.32; (d) 50 rad s^{-1} [← Problem]

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- OpenStax, *Calculus Volume 3*, by Gilbert Strang, Edwin "Jed" Herman et al. Licensed under CC BY 4.0. https://openstax.org/details/books/calculus-volume-3

The signal-processing chapter also draws on the following book, which is not openly licensed and is cited rather than reproduced: Marina Bosi and Richard E. Goldberg, *Introduction to Digital Audio Coding and Standards*, Kluwer Academic Publishers, 2003. The approximation of the threshold in quiet used there is taken from that book; the plot of it is our own.

The Fourier problems at the end of the Fourier chapter were written for this course by the author.

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